

# Spinors in Weyl Geometry

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## Abstract

We consider the wave equation for spinors in  $\mathcal{D}$ -dimensional Weyl geometry. By appropriately coupling the Weyl vector  $\phi_\mu$  as well as the spin connection  $\omega_{\mu ab}$  to the spinor field, conformal invariance can be maintained. The one loop effective action generated by the coupling of the spinor field to an external gravitational field is computed in two dimensions. It is found to be identical to the effective action for the case of a scalar field propagating in two dimensions.

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## I. INTRODUCTION

The effective action for gravity generated by the coupling of an external gravitational field to quantum matter fields is of crucial importance in string theory [1,2]. The behaviour of this induced gravitational theory under a conformal transformation is of particular interest. In refs. [3,4], the induced gravitational action in the case of a scalar matter field propagating in a background Weyl geometry is examined. In this paper, we extend these considerations to the case in which the matter field is a spinor field.

In the next section we consider how to couple a spinor field to a background gravitational field in the case of Weyl geometry in  $\mathcal{D}$ -dimensions. We then compute the first Seely-Gilkey/DeWitt-Schwinger coefficient  $E_1(x, \mathcal{D})$ , needed to determine the one-loop effective action in two dimensions when using the formalism of ref. [5].

## II. SPINORS IN WEYL GEOMETRY

Weyl in 1918 [6,7] introduced the idea of a conformal transformation

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x) \quad (1)$$

on the metric tensor  $g_{\mu\nu}(x)$ . (We use the conventions of [8], except that we employ the metric with signature ( + + + ...+ ) in  $\mathcal{D}$  dimensions. ) In addition, he employed a vector field  $\phi_\mu(x)$  in order to preserve the conformal invariance of the theory; the transformation of  $\phi_\mu$  that accompanies that of (1) is

$$\phi_\mu \rightarrow \phi_\mu + \Omega^{-1}\partial_\mu\Omega. \quad (2)$$

As is explained in [6,8], the role of  $\phi_\mu$  is to ensure that the magnitude of a vector  $\xi^\alpha$  ( $l^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta$ ) transforms as

$$dl = (\phi_\alpha dx^\alpha)l \quad (3)$$

under a conformal transformation. This in turn implies that the symmetric connection  $\Gamma_{\beta\gamma}^\alpha$  is given by

$$\Gamma_{\beta\gamma}^\alpha = -\frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}) + \delta_\beta^\alpha\phi_\gamma + \delta_\gamma^\alpha\phi_\beta - g_{\beta\gamma}\phi^\alpha. \quad (4)$$

The curvature tensor  $R_{\nu\alpha\beta}^\mu$  is defined by

$$\xi^\mu_{;\alpha;\beta} - \xi^\mu_{;\beta;\alpha} = R_{\nu\alpha\beta}^\mu \xi^\nu \quad (5)$$

where

$$\xi^\mu_{;\nu} = \xi^\mu_{,\nu} - \Gamma_{\lambda\nu}^\mu \xi^\lambda \quad (6)$$

and hence

$$R_{\nu\alpha\beta}^\mu = -\Gamma_{\nu\alpha,\beta}^\mu + \Gamma_{\nu\beta,\alpha}^\mu + \Gamma_{\lambda\beta}^\mu \Gamma_{\nu\alpha}^\lambda - \Gamma_{\lambda\alpha}^\mu \Gamma_{\nu\beta}^\lambda. \quad (7)$$

We also find from these equations that

$$g^{\mu\nu}_{;\lambda} = -2g^{\mu\nu}\phi_\lambda, \quad (8)$$

so that

$$g_{\mu\nu;\lambda} = +2g_{\mu\nu}\phi_\lambda. \quad (9)$$

Consequently, we find that

$$\begin{aligned} \xi^\mu_{;\mu} &= \frac{1}{\sqrt{g}}(\sqrt{g}\xi^\mu)_{,\mu} - \mathcal{D}\phi_\mu \xi^\mu \\ &= g^{\mu\nu}(\xi_{\mu;\nu}) - (\mathcal{D} - 2)\phi_\mu \xi^\mu \end{aligned} \quad (10)$$

and therefore

$$\xi_{\alpha;\beta;\gamma} - \xi_{\alpha;\gamma;\beta} = R_{\alpha\eta\beta\gamma}\xi^\eta + 2\xi_\alpha(\phi_{\beta;\gamma} - \phi_{\gamma;\beta}). \quad (11)$$

Since  $\psi_{;\beta;\gamma} - \psi_{;\gamma;\beta} = 0$ , we see that if  $\psi = g_{\tau\lambda}\xi^\tau\xi^\lambda$  then

$$\xi^\alpha\xi^\eta R_{\alpha\eta\beta\gamma} + \xi^2(\phi_{\beta;\gamma} - \phi_{\gamma;\beta}) = 0 \quad (12)$$

and hence  $R_{\alpha\eta\beta\gamma}$  is not anti-symmetric in the indices  $\alpha$  and  $\eta$ .

We now consider the vierbein field  $e_{\mu a}$  defined so that

$$g_{\mu\nu} = \eta^{ab} e_{\mu a} e_{\nu b} \quad (13)$$

and

$$\eta_{ab} = g^{\mu\nu} e_{\mu a} e_{\nu b}. \quad (14)$$

By (8) and (9), we see that

$$e^{\mu m}_{;\lambda} = -e^{\mu m} \phi_\lambda \quad (15)$$

and

$$e_{\mu m;\lambda} = e_{\mu m} \phi_\lambda. \quad (16)$$

These equations, when combined with the definition of the covariant derivative of  $e^{\mu m}$ ,

$$e^{\mu m}_{;\lambda} = e^{\mu m}_{,\lambda} - \Gamma_{\kappa\lambda}^\mu e^{\kappa m} + \omega_\lambda^m{}_n e^{\mu n} \quad (17)$$

show that the spin connection is given by

$$\begin{aligned} \omega_{\lambda m a} = & -\frac{1}{2} \left( e^\kappa_{m,\lambda} e_{\kappa a} - e^\kappa_{a,\lambda} e_{\kappa m} \right) - \frac{1}{2} e^\kappa_m e_{\lambda a,\kappa} + \frac{1}{2} e^\kappa_a e_{\lambda m,\kappa} \\ & + \frac{1}{2} g_{\pi\lambda} \left( e^\kappa_m e^\pi_{a,\kappa} - e^\kappa_a e^\pi_{m,\kappa} \right) - \phi_a e_{\lambda m} + \phi_m e_{\lambda a}. \end{aligned} \quad (18)$$

Under the conformal transformations of (1) and (2), we find that

$$e_{\mu a} \rightarrow \Omega e_{\mu a} \quad (19)$$

$$\Gamma_{\beta\gamma}^\alpha \rightarrow \Gamma_{\beta\gamma}^\alpha \quad (20)$$

$$\omega_{\mu mn} \rightarrow \omega_{\mu mn} \quad (21)$$

and

$$R \rightarrow \Omega^{-2} R \quad (22)$$

where  $R = g^{\mu\nu} R_{\mu\nu}$  and  $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ .

A conformally invariant coupling between the gravitational field defined by  $\phi_\mu$  and  $g_{\mu\nu}$  and a spinor field  $\psi$  is given by

$$S = \int d^D x e \bar{\psi} e^{c\mu} \left[ i \gamma_c \left( \partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \frac{D-1}{2} \phi_\mu \right) \right] \psi \quad (23)$$

where  $\gamma_c$  is a set of  $D$ -dimensional Dirac matrices satisfying  $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ , and  $\sigma_{ab} = (1/4) [\gamma_a, \gamma_b]$ . The field  $\psi$  undergoes the conformal transformation

$$\psi \rightarrow \Omega^{-\left(\frac{D-1}{2}\right)} \psi \quad (24)$$

in conjunction with (1) and (2). A term of the form  $e \bar{\psi} \sqrt{R} \psi$  in (23) would also be conformally invariant, but will not be considered due to its non-polynomial dependence on  $R$ .

At one-loop order, the effective action for gravity due to the propagation of the spinor field  $\psi$  in (23) is given by  $\det \mathcal{D}$  where

$$\mathcal{D} = e^{c\mu} \gamma_c \left( \partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \frac{D-1}{2} \phi_\mu \right). \quad (25)$$

Since  $\mathcal{D}$  is linear in the derivatives, we will replace  $\det \mathcal{D}$  by  $\det^{1/2} (\mathcal{D}^2)$ . Despite the fact that  $\mathcal{D}$  is not Hermitian due to the term proportional to  $\phi_\mu$  in (23), the anti-Hermitian part of  $\mathcal{D}^2$  does not contribute in two dimensions as is demonstrated by the following calculation in which  $\mathcal{D}^2$  is simplified.

If  $\gamma_\mu = e_{\mu c} \gamma^c$  and  $D_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \frac{D-1}{2} \phi_\mu$ , then by (25),

$$\mathcal{D}^2 = \gamma^\mu ([D_\mu, \gamma^\nu] + \gamma^\nu D_\mu) D_\nu \quad (26)$$

with

$$[D_\mu, \gamma^\nu] = e^\nu_{a,\mu} \gamma^a + \frac{1}{2} \omega_{\mu ab} e^\nu_c [\sigma^{ab}, \gamma^c] \quad (27)$$

where in both four and two dimensions

$$[\sigma_{ab}, \gamma^c] = \gamma_a \delta_b^c - \gamma_b \delta_a^c. \quad (28)$$

Together, (15), (17) and (27) yield

$$[D_\mu, \gamma^\nu] = [\Gamma_{\mu\kappa}^\nu e_p^\kappa - e_p^\nu \phi_\mu] \gamma^p \quad (29)$$

so that

$$\not{D}^2 = [e_q^\mu (\delta^{qp} + 2\sigma^{qp}) (\Gamma_{\mu\kappa}^\nu e_p^\kappa - e_p^\nu \phi_\mu) + (g^{\mu\nu} + 2\sigma^{\mu\nu}) D_\mu] D_\nu. \quad (30)$$

We also can use the relation

$$e_q^\mu e_p^\kappa \delta^{qp} \Gamma_{\mu\kappa}^\nu = g^{\mu\kappa} \Gamma_{\mu\kappa}^\nu$$

which by (4) becomes

$$= g^{\nu\sigma}{}_{,\sigma} + \frac{1}{2} g^{\nu\lambda} g^{\mu\sigma} g_{\mu\sigma,\lambda} + (2 - \mathcal{D}) \phi^\nu. \quad (31)$$

Consequently as

$$\frac{1}{2} g^{\mu\sigma} g_{\mu\sigma,\lambda} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^\lambda} \quad (32)$$

we have

$$\not{D}^2 = \left[ \frac{1}{\sqrt{g}} (\sqrt{g} g^{\nu\lambda})_{,\lambda} + (1 - \mathcal{D}) \phi^\nu - 2\sigma^{qp} e_q^\mu e_p^\nu \phi_\mu + (g^{\mu\nu} + 2\sigma^{\mu\nu}) D_\mu \right] D_\nu. \quad (33)$$

We now find that as

$$\begin{aligned} \sigma^{\mu\nu} D_\mu D_\nu &= \frac{1}{2} \sigma^{\mu\nu} [D_\mu, D_\nu] \\ &= \frac{1}{2} \sigma^{\mu\nu} \left[ \frac{\mathcal{D}-1}{2} (\phi_{\nu,\mu} - \phi_{\mu,\nu}) + \frac{1}{2} \sigma^{ab} (\omega_{\nu ab,\mu} - \omega_{\mu ab,\nu}) + \frac{1}{4} \omega_{\mu ab} \omega_{\nu cd} [\sigma^{ab}, \sigma^{cd}] \right] \end{aligned} \quad (34)$$

with

$$[\sigma_{ab}, \sigma_{cd}] = -[\eta_{ac} \sigma_{bd} - \eta_{ad} \sigma_{bc} + \eta_{bd} \sigma_{ac} - \eta_{bc} \sigma_{ad}],$$

( and, in four dimensions  $\{\sigma_{ab}, \sigma_{cd}\} = \frac{1}{2} [-\eta_{ac} \eta_{bd} + \eta_{ad} \eta_{bc} + \epsilon_{abcd} \gamma_5]$  ) we have

$$\sigma^{\mu\nu} D_\mu D_\nu = \frac{1}{2} \sigma^{\mu\nu} \left[ \left( \frac{\mathcal{D}-1}{2} \right) F_{\mu\nu} - \frac{1}{2} \sigma^{ab} R_{\mu\nu ab} \right]. \quad (35)$$

This involves use of the relations  $R_{\mu\nu ab} = -\omega_{\nu ab,\mu} + \omega_{\mu ab,\nu} - \omega_{\mu am} \omega_\nu^m{}_b + \omega_{\nu am} \omega_\mu^m{}_b$  with  $R_{\mu\nu ab} = e_a^\alpha e_b^\beta R_{\mu\nu\alpha\beta}$  and  $F_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}$ .

We also have

$$\begin{aligned}
-2\sigma^{qp}e^\mu{}_qe^\nu{}_p\phi_\mu D_\nu &= 2A^\nu \dot{D}_\nu \\
&= \frac{1}{\sqrt{g}} \left[ (\dot{D}_\mu + A_\mu) \sqrt{g} g^{\mu\nu} (\dot{D}_\nu + A_\nu) - \dot{D}_\mu (\sqrt{g} g^{\mu\nu}) \dot{D}_\nu \right. \\
&\quad \left. - (\dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu) - \sqrt{g} A_\mu A^\mu \right]
\end{aligned} \tag{36}$$

where

$$A^\mu = \sigma^{\mu\nu} \phi_\nu \tag{37}$$

and

$$\dot{D}_\mu \equiv \partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab}, \tag{38}$$

( so that  $(\dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu) = (\sqrt{g} g^{\mu\nu} A_\nu)_{,\mu} + \frac{1}{2} \sqrt{g} g^{\mu\nu} [\omega_{\mu ab} \sigma^{ab}, A_\nu]$  ) as well as

$$\begin{aligned}
\left[ \frac{1}{\sqrt{g}} (g^{\mu\nu} \sqrt{g})_{,\mu} + (1 - \mathcal{D}) \phi^\nu + g^{\mu\nu} D_\mu \right] D_\nu &= \frac{1}{\sqrt{g}} \dot{D}_\mu (\sqrt{g} g^{\mu\nu}) \dot{D}_\nu - \left( \frac{\mathcal{D}-1}{2} \right)^2 \phi_\nu \phi^\nu \\
&\quad + \left( \frac{\mathcal{D}-1}{2} \right) \frac{1}{\sqrt{g}} (\sqrt{g} g^{\mu\nu} \phi_\mu)_{,\nu}.
\end{aligned} \tag{39}$$

Together (30),(35) and (36) reduce (26) to

$$\begin{aligned}
\mathcal{D}^2 &= \frac{1}{\sqrt{g}} \left[ (\dot{D}_\mu + A_\mu) \sqrt{g} g^{\mu\nu} (\dot{D}_\nu + A_\nu) - (\dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu) - \sqrt{g} A_\mu A^\mu - \left( \frac{\mathcal{D}-1}{2} \right)^2 \sqrt{g} \phi_\mu \phi^\mu \right. \\
&\quad \left. + \frac{\mathcal{D}-1}{2} (\sqrt{g} g^{\mu\nu} \phi_\mu)_{,\nu} \right] + \sigma^{\mu\nu} \left[ \left( \frac{\mathcal{D}-1}{2} \right) F_{\mu\nu} - \frac{1}{2} \sigma^{ab} R_{\mu\nu ab} \right].
\end{aligned} \tag{40}$$

We note that  $\mathcal{D}^2$  is not Hermitian due to the presence of the terms  $-(\dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu) + \frac{\mathcal{D}-1}{2} \sigma_{\mu\nu} F^{\mu\nu}$  in (40). However, as we shall show in the next section, these terms do not contribute to the effective action in two dimensions.

### III. THE EFFECTIVE ACTION FOR GRAVITY

In order to compute the effective action for gravity induced by the propagation of a spinor in two-dimensional Weyl geometry, we use the formalism of ref. [5]. In this approach, we consider the operator

$$\mathbb{A} = e^{1/2} (i\mathbb{D}) e^{-1/2} \quad (41)$$

where  $\mathbb{D}$  is defined in (25). If we make the transforms of eqs. (1), (2), (19 - 22) so that  $\mathbb{A} \rightarrow \bar{\mathbb{A}}$  then it is easy to see that in all dimensions

$$\mathbb{A} = \Omega^{1/2} \bar{\mathbb{A}} \Omega^{1/2} \quad (42)$$

and thus the formalism used to determine effective action for spinors propagating in a Riemannian background in ref. [5] can be used in the case of a Weyl background; the one difference being that the expansion coefficient  $E_1(x)$  given in (A21) must be used in conjunction with the explicit forms for  $V_\mu$  and  $X$  that occur in (40), viz.

$$V_\mu = \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \sigma_{\mu\nu} \phi^\nu \quad (43)$$

and

$$X = \frac{1}{\sqrt{g}} \left[ \left( \dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu \right) + \sqrt{g} A_\mu A^\mu + \frac{1}{4} \sqrt{g} \phi_\mu \phi^\mu - \frac{1}{2} (\sqrt{g} g^{\mu\nu} \phi_\nu)_{,\mu} \right] - \frac{1}{2} \sigma^{\mu\nu} (F_{\mu\nu} - \sigma^{ab} R_{\mu\nu ab}). \quad (44)$$

Since in two dimensions  $\sigma_{\mu\nu} = \frac{1}{2i} \epsilon_{\mu\nu} \gamma_5$ ,  $tr(\sigma^{\mu\nu}) = 0$  and  $tr(\sigma_{\mu\nu} \sigma_{\alpha\beta}) = -\frac{1}{2} (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha})$  we see that

$$\begin{aligned} tr(E_1(x)) &= \frac{1}{4\pi} tr \left[ 2 \left( \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \sigma_{\mu\nu} \phi^\nu \right) \phi^\mu - \frac{1}{\sqrt{g}} \left[ \left( \dot{D}_\mu \sqrt{g} g^{\mu\nu} A_\nu \right) + \sqrt{g} A_\mu A^\mu + \frac{1}{4} \sqrt{g} \phi_\mu \phi^\mu \right. \right. \\ &\quad \left. \left. - \frac{1}{2} (\sqrt{g} g^{\mu\nu} \phi_\nu)_{,\mu} \right] + \frac{1}{2} \sigma^{\mu\nu} (F_{\mu\nu} - \sigma^{ab} R_{\mu\nu ab}) - \frac{1}{3} g^{\alpha\beta} \phi_{\alpha;\beta} - \frac{1}{6} R \right] \\ &= \frac{1}{24\pi} [R + 2g^{\alpha\beta} (\phi_\alpha)_{;\beta}]. \end{aligned} \quad (45)$$

Since in  $\mathcal{D}$  dimensions [6,8]  $R = \dot{R} + (\mathcal{D} - 1)(\mathcal{D} - 2)\phi_\alpha \phi^\alpha - 2(\mathcal{D} - 1)\frac{1}{\sqrt{g}}(\sqrt{g}\phi^\alpha)_{,\alpha}$  where  $\dot{R}$  is the Ricci scalar in the limit  $\phi_\alpha \rightarrow 0$ ,  $tr(E_1) = \frac{1}{24\pi}\dot{R}$ , which is the Riemannian space limit. (Those contributions to  $X$  in (44) that come from the non-Hermitian parts of  $H$  have vanishing trace and hence do not contribute to the effective action, as was noted at the end of the preceding section.)

The result of (45) is precisely the same as the result that one obtains for a model in which a scalar propagates in a two-dimensional space with Weyl geometry [3,4]. Consequently, the



arguments of [5] can be used in the same way that they were in the scalar case, and we obtain

$$S_{eff} = \frac{1}{24\pi} \int d^2x \sqrt{g} \left[ -\sigma \left( R + \frac{2}{\sqrt{g}} (\sqrt{g}\phi^\lambda)_{,\lambda} \right) + \sigma \square \sigma \right] \quad (46)$$

(where  $\sigma = \ln(\Omega^{-1})$ ). The integrand in (46) reduces to what occurs in Riemannian space in two dimensions as was argued above. Hence in two dimensions, the effective action for gravity induced by the propagation of a spinor in a background with Weyl geometry is identical to what one obtains with Riemannian background.

#### IV. DISCUSSION

We have analyzed in some detail the formalism appropriate for a spinor field propagating in a space-time with Weyl geometry. Despite the fact that the action for the spinor field is not Hermitian, we have shown that the anti-Hermitian contribution does not contribute to the effective action for gravity in two dimensions that is induced by the propagation of the spinor field. Indeed, the spinor contribution turns out to be precisely equal to that of the scalar field in two dimensions.

Currently we are considering expanding our considerations to include a version of supergravity that involves Weyl geometry in the bosonic sector.

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#### APPENDIX A:

In this appendix we consider the matrix element

$$M_{xy} = \langle x | e^{-Ht} | y \rangle \quad (\text{A1})$$

for the operator

$$H = -\frac{1}{\sqrt{g}} (\partial_\mu + V_\mu) \sqrt{g} g^{\mu\nu} (\partial_\nu + V_\nu) + X. \quad (\text{A2})$$

In order to compute the coefficient  $E_1(x)$  in the expansion [9]

$$M_{xx} = \frac{1}{t^{D/2}} [E_0(x) + E_1(x)t + E_2(x)t^2 + \dots] \quad (\text{A3})$$

we employ the approach of [10,11] in conjunction with a normal coordinate expansion, as was employed in the non-linear sigma model [12].

We begin by expanding the metric tensor in normal coordinate about a point  $x$  to the order that will eventually be required to obtain  $E_1(x)$ ,

$$g_{\mu\nu}(x + \pi(\xi)) = g_{\mu\nu} + g_{\mu\nu;\lambda} \xi^\lambda + \frac{1}{2} \left[ g_{\mu\nu;\lambda\sigma} + \frac{1}{3} (R^\kappa_{\lambda\mu\sigma} g_{\kappa\nu} + R^\kappa_{\lambda\nu\sigma} g_{\kappa\mu}) \right] \xi^\lambda \xi^\sigma + \dots \quad (\text{A4})$$

which, by (8), becomes

$$= g_{\mu\nu} + 2g_{\mu\nu} \phi_\lambda \xi^\lambda + \frac{1}{2} \left[ 4g_{\mu\nu} \phi_\lambda \phi_\sigma + g_{\mu\nu} (\phi_{\lambda;\sigma} + \phi_{\sigma;\lambda}) + \frac{1}{3} (R^\kappa_{\lambda\mu\sigma} g_{\kappa\nu} + R^\kappa_{\lambda\nu\sigma} g_{\kappa\mu}) \right] \xi^\lambda \xi^\sigma + \dots \quad (\text{A5})$$

Consequently, we find that

$$\begin{aligned} \sqrt{g(x + \pi(\xi))} &= \exp \left[ \frac{1}{2} \text{tr} [\ln(g_{\mu\nu})] \right] \\ &= \sqrt{g} \left[ 1 + \mathcal{D}\phi \cdot \xi + \frac{\mathcal{D}^2}{2} (\phi \cdot \xi)^2 + \frac{\mathcal{D}}{2} \phi_{\lambda;\sigma} \xi^\lambda \xi^\sigma + \frac{1}{6} R_{\lambda\sigma} \xi^\lambda \xi^\sigma + \dots \right] \end{aligned} \quad (\text{A6})$$

and so

$$\frac{1}{\sqrt{g(x + \pi(\xi))}} = \frac{1}{\sqrt{g}} \left[ 1 - \mathcal{D}\phi \cdot \xi + \frac{\mathcal{D}^2}{2} (\phi \cdot \xi)^2 - \frac{\mathcal{D}}{2} \phi_{\lambda;\sigma} \xi^\lambda \xi^\sigma - \frac{1}{6} R_{\lambda\sigma} \xi^\lambda \xi^\sigma + \dots \right]. \quad (\text{A7})$$

We also will have occasion to use the expansions

$$V_\mu(x + \pi(\xi)) = V_\mu + V_{\mu;\lambda} \xi^\lambda + \frac{1}{2!} \left( V_{\mu;\lambda;\sigma} + \frac{1}{3} R^\kappa_{\lambda\mu\sigma} V_\kappa \right) \xi^\lambda \xi^\sigma + \dots \quad (\text{A8})$$

and

$$X(x + \pi(\xi)) = X + X_{;\lambda} \xi^\lambda + \frac{1}{2!} X_{;\lambda;\sigma} \xi^\lambda \xi^\sigma + \dots \quad (\text{A9})$$

For purpose of illustration, we now let  $X = V_\mu = 0$  in (A2), so that

$$\begin{aligned} H &= -\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \\ &= \left(1 + Y_\lambda \xi^\lambda + Y_{\lambda\sigma} \xi^\lambda \xi^\sigma\right) p_\mu \left(\delta^{\mu\nu} + Z_\lambda^{\mu\nu} \xi^\lambda + Z_{\lambda\sigma}^{\mu\nu} \xi^\lambda \xi^\sigma\right) p_\nu \end{aligned} \quad (\text{A10})$$

to the order which we need to compute  $E_1(x)$ . (Here we have defined  $p$  to be  $-i\partial$  and have assumed  $g_{\mu\nu}(x) = \delta_{\mu\nu}$ .) In order to expand  $M_{xx}$  in (A1) in powers of  $t$  as in (A3), we follow the approach of [10,11]. This involves first using the Schwinger expansion [13]

$$\begin{aligned} e^{-(H_0+H_1)t} &= e^{-H_0 t} + (-t) \int_0^1 du e^{-(1-u)H_0 t} H_1 e^{-uH_0 t} \\ &\quad + (-t)^2 \int_0^1 du u \int_0^1 dv e^{-(1-u)H_0 t} H_1 e^{-u(1-v)H_0 t} H_1 e^{-uvH_0 t} + \dots \end{aligned} \quad (\text{A11})$$

where  $H_0$  is independent of  $\xi^\lambda$ , and then converting this into a power series in  $t$  by insertion of complete sets of momentum states and then performing the appropriate momentum integrals.

Explicitly, we find that to the order that contributes to  $E_1$ ,

$$\begin{aligned} M_{xx} &\approx \langle x | \left[ (-t) \int_0^1 du e^{-(1-u)p^2 t} \left( Y_{\alpha\beta} \xi^\alpha \xi^\beta p^2 + p_\mu Z_{\alpha\beta}^{\mu\nu} \xi^\alpha \xi^\beta p_\nu + Y_\alpha \xi^\alpha p_\mu Z_\beta^{\mu\nu} \xi^\beta p_\nu \right) e^{-up^2 t} \right. \\ &\quad \left. + (-t)^2 \int_0^1 du u \int_0^1 dv e^{-(1-u)p^2 t} \left( Y_\alpha \xi^\alpha p^2 + p_\mu Z_\alpha^{\mu\nu} \xi^\alpha p_\nu \right) e^{-u(1-v)p^2 t} \right. \\ &\quad \left. \times \left( Y_\beta \xi^\beta p^2 + p_\lambda Z_\beta^{\lambda\sigma} \xi^\beta p_\sigma \right) e^{-uvp^2 t} + \dots \right] | x \rangle. \end{aligned} \quad (\text{A12})$$

If we now insert complete sets of position and momentum states into (A12) we obtain

$$\begin{aligned} M_{xx} &\approx (-t) \int_0^1 du \int \frac{dp dq}{(2\pi)^{2\mathcal{D}}} \int dz e^{-i(p-q) \cdot z} e^{-[(1-u)p^2 + uq^2]t} \left( Y_{\alpha\beta} z^\alpha z^\beta q^2 + p_\mu Z_{\alpha\beta}^{\mu\nu} z^\alpha z^\beta q_\nu \right) \\ &\quad + (-t) \int_0^1 du \int \frac{dp dq dr}{(2\pi)^{3\mathcal{D}}} \int dz_1 dz_2 e^{-i[(p-q) \cdot z_1 + (q-r) \cdot z_2]} e^{-[(1-u)p^2 + ur^2]t} \left( Y_\alpha z_1^\alpha q_\mu Z_\beta^{\mu\nu} z_2^\beta r_\nu \right) \\ &\quad + (-t)^2 \int_0^1 du u \int_0^1 dv \int \frac{dp dq dr}{(2\pi)^{3\mathcal{D}}} \int dz_1 dz_2 e^{-i[(p-q) \cdot z_1 + (q-r) \cdot z_2]} e^{-[(1-u)p^2 + u(1-v)q^2 + uvr^2]t} \\ &\quad \times \left( Y_\alpha z_1^\alpha q^2 + p_\mu Z_\alpha^{\mu\nu} z_1^\alpha q_\nu \right) \left( Y_\beta z_2^\beta r^2 + q_\lambda Z_\beta^{\lambda\sigma} z_2^\beta r_\sigma \right). \end{aligned} \quad (\text{A13})$$

It is now a straightforward exercise to first integrate over  $z_i$  in (A13), and then after utilizing the resulting delta functions, evaluate the momentum integrals over  $p, q$  and  $r$  (all of which are Gaussian). We finally arrive the result

$$\begin{aligned}
M_{xx} \approx & \frac{t}{(4\pi t)^{\mathcal{D}/2}} \left[ Y_{\alpha\alpha} \left( -\frac{\mathcal{D}}{6} + \frac{2}{3} \right) + \left( -\frac{1}{6} Z_{\nu\nu}^{\mu\mu} - \frac{1}{3} Z_{\mu\nu}^{\mu\nu} \right) \right. \\
& + Y_{\alpha} Y_{\alpha} (\mathcal{D} + 2) \left( \frac{\mathcal{D} - 4}{48} \right) + Y_{\mu} Z_{\nu}^{\mu\nu} \left( -\frac{\mathcal{D}}{12} + \frac{1}{3} \right) + Y_{\mu} Z_{\mu}^{\nu\nu} \left( \frac{\mathcal{D} - 4}{24} \right) \\
& \left. + \left( \frac{1}{24} Z_{\alpha}^{\mu\nu} Z_{\alpha}^{\mu\nu} + \frac{1}{12} Z_{\lambda}^{\mu\nu} Z_{\nu}^{\mu\lambda} + \frac{1}{48} Z_{\lambda}^{\mu\mu} Z_{\lambda}^{\nu\nu} - \frac{1}{12} Z_{\mu}^{\mu\lambda} Z_{\lambda}^{\nu\nu} \right) \right]. \quad (\text{A14})
\end{aligned}$$

If in (A10) we were to include  $X$  and  $V_{\mu}$  as in (A2), then by employing the expansions of (A8) and (A9), we find that in order to determine  $E_1(x)$ , (A14) must be supplemented by

$$M_{xx} \approx \frac{t}{(4\pi t)^{\mathcal{D}/2}} (-V_{\nu} Y^{\nu} - X). \quad (\text{A15})$$

It is now possible to determine  $E_1$  for the particular case in which  $Y_{\alpha}$ ,  $Y_{\alpha\beta}$ ,  $Z_{\alpha}^{\mu\nu}$  and  $Z_{\alpha\beta}^{\mu\nu}$  are fixed by the expansions (A5 - A9). We find that these imply that

$$Y_{\alpha} = -\mathcal{D}\phi_{\alpha} \quad (\text{A16})$$

$$Y_{\alpha\beta} = \frac{\mathcal{D}^2}{2}\phi_{\alpha}\phi_{\beta} - \frac{\mathcal{D}}{4}(\phi_{\alpha;\beta} + \phi_{\beta;\alpha}) - \frac{1}{6}R_{\alpha\beta} \quad (\text{A17})$$

$$Z_{\alpha}^{\mu\nu} = (\mathcal{D} - 2)g^{\mu\nu}\phi_{\alpha} \quad (\text{A18})$$

and

$$\begin{aligned}
Z_{\alpha\beta}^{\mu\nu} = & \left[ g^{\mu\nu} \left( \frac{\mathcal{D}^2}{2} - 2\mathcal{D} + 2 \right) \phi_{\alpha}\phi_{\beta} + g^{\mu\nu} \left( \frac{\mathcal{D}}{4} - \frac{1}{2} \right) (\phi_{\alpha;\beta} + \phi_{\beta;\alpha}) \right. \\
& \left. + \frac{1}{6}g^{\mu\nu}R_{\alpha\beta} - \frac{1}{6}(R_{\alpha}^{\mu}{}^{\nu}{}_{\beta} + R_{\alpha}^{\nu}{}^{\mu}{}_{\beta}) \right]. \quad (\text{A19})
\end{aligned}$$

Together, (A14 - A18) imply that when  $\mathcal{D} = 2$ ,

$$M_{xx} \approx \frac{t}{(4\pi t)} \left( 2V_{\mu}\phi^{\mu} - X - \frac{1}{3}g^{\alpha\beta}\phi_{\alpha;\beta} - \frac{1}{6}R \right). \quad (\text{A20})$$

By (A20) we see that in two dimensions

$$4\pi E_1(x) = 2V_{\mu}\phi^{\mu} - X - \frac{1}{3}g^{\alpha\beta}\phi_{\alpha;\beta} - \frac{1}{6}R \quad (\text{A21})$$

when considering the operator  $H$  in (A2).

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